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ON MULTIPLE IMPACT†

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The relationships and paradoxes of the problem of multiple impact are discussed. The latter includes not only the case of simultaneous collision between three or more bodies, but also problems involving a collision between two bodies when there are additional constraints. By solving a number of problems, it is shown that the following kinds of multiple impact can be distinguished depending on the configuration of the system and the dynamical properties of the colliding bodies.

1. The regular type is characterized by the fact that the problem is correctly solvable within the framework of the given mechanical system with a finite number of degrees of freedom. In this case small variations of the initial conditions lead to small modifications of the same order of magnitude of the velocities after the collision.

2. The stochastic type combines high sensitivity of the result to the initial conditions with the impossibility of determining these conditions with sufficient accuracy. In this case it appears that one should consider the impact impulse as a random variable with a discrete set of values.

3. In the quasiregular case the problem under consideration is solvable, but the solution depends very much on the physical properties of the colliding bodies. To obtain this solution it is no longer sufficient to consider a finite-dimensional mechanical system.

Regularity criteria for a collision between three or more free bodies and for the impact of a physical pendulum against an obstacle are obtained.

The problem of multiple impact dates back to the eighteenth century. Bernoulli [1] studied an absolutely elastic collision in a symmetric system of spheres. MacLaurin used the Newtonian coefficients of restitution [2] to describe a multiple impact. D'Alembert interpreted impact impulses as resulting from elastic deformations and arrived at the unexpected conclusion that the result obtained in this approach differs from the sum of impulses computed separately for each pair [3]. D'Alembert's argument did not receive due recognition, and reduction to pairwise collisions has been used up until now to solve the problem of multiple impact. The nature of the paradoxes of this approach was discussed in [4, 5].

1. METHODS OF SOLVING THE MULTIPLE-IMPACT PROBLEM

In dynamics a collision between rigid bodies is considered as their short-term interaction leading to a sudden change in the velocities. As we know, even the simplest problem involving a direct central collision between two non-rotating spheres cannot be solved without certain additional physical assumptions. Newton's hypothesis, according to which the velocities $v_{1,2}$ and $V_{1,2}$ before and after the collision are related by the equation

$$V_1 - V_2 = \kappa(v_2 - v_1), \quad \kappa \in [0,1] \tag{1.1}$$

is used most frequently. Here κ is the coefficient of restitution. The limit values $\kappa = 1$ and $\kappa = 0$ correspond to absolutely elastic and plastic impact, respectively.

Momentum conservation serves as another condition which can be used to determine the two unknowns $V_{1,2}$

$$\sum_{j=1}^{2} m_j V_j = \sum_{j=1}^{2} m_j v_j \tag{1.2}$$

where $m_{1,2}$ are the masses of the spheres.

The system (1.1), (1.2) has a unique solution, which also appears to be realistic, even though precise experiments indicate that the coefficient κ depends on the relative velocity $v_1 - v_2$ of the spheres as they approach each other [6].

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This approach can be extended to the case of an off-centre impact of spheres as well as rigid bodies of arbitrary shape with smooth convex surfaces. To this end one must set up equations similar to (1.1) for the normal velocity components at the point of contact and take into account that the tangential components are not altered when there is no friction.

It is often useful to represent an impact geometrically as a reflection of a representative point by the boundary of the domain of existence in the configuration space of the system [4, 5, 7]. To this end we express the kinetic energy in terms of the generalized velocities

$$T = \tfrac{1}{2}\dot{\mathbf{q}}\mathbf{A}(\mathbf{q})\dot{\mathbf{q}}^{\mathrm{T}}, \quad \mathbf{q} = (q_1, q_2, \ldots, q_n) \tag{1.3}$$

and define a scalar product in the tangent space $TM_{\mathbf{q}}$ using the matrix $\mathbf{A}(\mathbf{q})$

$$(\mathbf{u}_1, \mathbf{u}_2)_{\mathbf{q}} = \mathbf{u}_1\mathbf{A}(\mathbf{q})\mathbf{u}_2^{\mathrm{T}} \tag{1.4}$$

The domain of admissible coordinate values can be represented as the family of solutions of the inequality

$$f(\mathbf{q}) \geqslant 0 \tag{1.5}$$

where equality corresponds to the bodies in contact with one another and strict inequality indicates that there is no contact. A normal vector to the surface $f(\mathbf{q}) = 0$ in the sense of the Euclidean structure (1.4) can be given as follows:

$$\mathbf{n} = \operatorname{grad} f(\mathbf{q})\mathbf{A}^{-1}(\mathbf{q}) \tag{1.6}$$

If the representative point hits the impact surface in the direction of $\mathbf{n}$, then it is reflected in the same direction

$$\dot{\mathbf{q}}_n^+ = -\kappa\dot{\mathbf{q}}_n^- \tag{1.7}$$

The velocity components parallel to the surface $f(\mathbf{q}) = 0$ remain unchanged for an impact without friction.

The simultaneous collision of three or more bodies corresponds to the case when the representative point hits an edge of the domain of existence, which is the intersection of several smooth surfaces

$$f_1(\mathbf{q}) \geqslant 0, \ f_2(\mathbf{q}) \geqslant 0, \ \ldots, \ f_k(\mathbf{q}) \geqslant 0 \tag{1.8}$$

The following are the most widely used methods of solving the multiple impact problem.

1. The method of independent restitution [2] consists of reducing a multiple impact to the sum of "pairwise" collisions, each described by a condition of type (1.7). The k equalities characterizing the restitution of each of the unilateral constraints are supplemented by $n - k$ conditions for the conservation of the tangential components of the velocity.

2. In the method of successive impacts a multiple impact is represented as a sequence of successive impacts of the system against the unilateral constraints (1.8).

3. The method of indentation is based on using deformations in the colliding bodies, which gives rise to impact forces [3]. As we know [6, 8, 9], this approach is more realistic when applied to a collision between two bodies than Newton's hypothesis (1.1).

The advantages and disadvantages of the above-mentioned methods will be discussed below. Here we will mention one more approach, typical for problems in statistical mechanics. In impact theory it is apparently used for the first time.

4. The statistical ensemble method is applied in those cases when the errors in determining the coordinates of the system cannot be regarded as negligible because they have a significant effect on the evolution of the system. One must consider a collection of identical systems with different initial conditions, the result of an impact being defined as a random variable.

We will now discuss specific systems.

2. THE COLLINEAR COLLISION OF THREE SPHERES

We will consider the problem of simultaneous collision between three spheres of mass m_1, m_2, m_3, which is the simplest problem involving a multiple impact. The simplicity of the setting enables us to verify various hypotheses experimentally: one only needs a few coins on a smooth table surface.

The only indisputable equation for finding the three unknowns V_{1-3} is similar to (1.2)

$$\sum_{j=1}^{3} m_j V_j = \sum_{j=1}^{3} m_j v_j \tag{2.1}$$

Some additional hypotheses are therefore necessary to solve the problem. Let us consider the methods presented in the previous section.

1. The independent restitution method consists of specifying the coefficients of restitution for each of the two pairs 1–2 and 2–3 (Fig. 1) by analogy with (1.1)

$$V_1 - V_2 = \kappa_{12}(v_2 - v_1), \quad V_2 - V_3 = \kappa_{23}(v_3 - v_2) \tag{2.2}$$

System (2.1), (2.2) has a unique solution, but it is completely unrealistic. A simple experiment demonstrates this for a system of three identical billiard balls, two of which are initially stationary and are hit by the third one ($v_1 > 0$, $v_2 = v_3 = 0$). As a result, the ball on the opposite side to the hitting ball will move away, i.e. $V_1 = V_2 = 0$, $V_3 > 0$. This would mean that $\kappa_{23} = \infty$, $\kappa_{12} = 0$ in (2.2), even though $\kappa_{12} = \kappa_{23}$, since the balls are identical.

One can draw the conclusion that independent coefficients of restitution are not applicable to the solution of the given problem. The case of plastic impact $\kappa_{12} = \kappa_{23} = 0$ is the only exception: in this case $V_1 = V_2 = V_3$. Below we shall consider elastic impacts only.

2. In the second approach the pairwise collisions do not occur simultaneously, but one after another. We begin the impact between the first and second balls, neglecting the existence of the third one. We use (1.2) and the first equation in (2.2). Then we proceed to the collision between the second and third balls in the absence of the first one, taking the velocity V_2' of the second ball after the collision with the first one, rather than the initial velocity v_2. This may not be the last collision. The sequence of collisions must be continued until $V_1 \leqslant V_2 \leqslant V_3$.

This approach looks more attractive than the previous one because it enables us to obtain a reasonable solution of the multiple impact problem. In the case of three identical balls with absolutely rigid impact and with the initial conditions $v_1 > 0$, $v_2 = v_3 = 0$ it leads to the following result: $V_1' = 0$, $V_2' = v_1$, $V_3' = 0$ after the collision between the first and second balls and $V_1 = V_2 = 0$, $V_3 = v_1$ after the collision between the second and third balls. This is consistent with experimental data.

Unfortunately, this successful agreement is not always the case, as one can see from the following example.

Example. Let the parameters of the system and the initial conditions be as follows:

$$m_1 = m_3 = \tfrac{1}{3} m_2, \quad \kappa_{12} = \kappa_{23} = 1, \quad v_1 = 1, \quad v_2 = 0, \quad v_3 = -1$$

There are two ways of realizing the method in question, leading to different results. In the first version the collision between the first and second balls is considered first, as a result of which $V_1' = -1/2$, $V_2' = 1/2$, $V_3' = -1$, followed by the collision between the second and third balls. As a result, $V_1 = -1/2$, $V_2 = -1/4$, $V_3 = 5/4$. In the second version it is assumed that the second and third balls collide first, so that $V_1' = 1$, $V_2' = -1/2$, $V_3' = 1/2$, followed by the first and second balls ($V_1 = -5/4$, $V_2 = 1/4$, $V_3 = 1/2$).

Despite the symmetry of the system with respect to the central (second) ball, neither of the solutions constructed is symmetric.

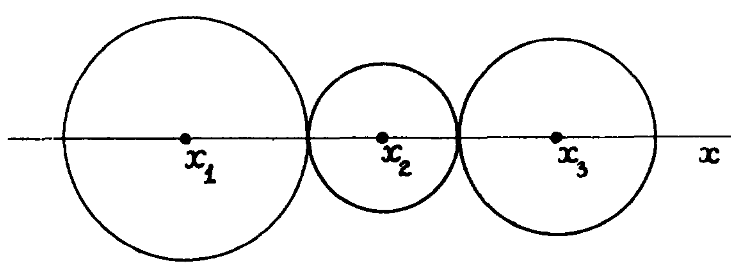


Fig. 1.

One can conclude that, in general, the method of consecutive "pairwise" collisions does not lead to a unique solution of the multiple impact problem.

3. To use the indentation method we introduce a system of coordinates on the straight line passing through the centres of the balls. We denote by x_j the coordinates of the centres and by ρ_j the radii of the balls ($j = 1, 2, 3$). If the balls were absolutely rigid, then the equalities

$$x_2 - x_1 = \rho_1 + \rho_2, \quad x_3 - x_3 = \rho_2 + \rho_3$$

would be satisfied for a multiple impact. The normal deformations can therefore be defined by

$$\zeta_{1,2} = \max\{0, \delta_{1,2}\}, \ \delta_1 = \rho_1 + \rho_2 - x_2 + x_1, \ \delta_2 = \rho_2 + \rho_3 - x_3 + x_2 \tag{2.3}$$

In the classical stereomechanical impact theory the accompanying vibrations are neglected, and so is the action of "finite" forces [6]. With these assumptions, the equations describing a collinear collision of three balls can be represented in the form

$$m_1[\dot{x}_1(t) - \dot{x}_1(t_0)] = -I_1 = -\int_{t_0}^{t} R_{12}dt \tag{2.4}$$

$$m_3[\dot{x}_3(t) - \dot{x}_3(t_0)] = I_2 = \int_{t_0}^{t} R_{23}dt \quad m_2[\dot{x}_2(t) - \dot{x}_2(t_0)] = I_1 - I_2$$

where t_0 and t are the initial and current instants of impact and $R_{12} = R_{12}(\zeta_1, \dot{\zeta}_1)$ and $R_{23} = R_{23}(\zeta_2, \dot{\zeta}_2)$ are the impact reactions (when there is no dissipation they depend only on ζ). Taking (2.3) into account, Eqs (2.4) can be written as follows in the domain $\delta_1 > 0$, $\delta_2 > 0$

$$\ddot{\delta}_1 = m_2^{-1} R_{23}(\delta_2, \dot{\delta}_2) - (m_1^{-1} + m_2^{-1}) R_{12}(\delta_1, \dot{\delta}_1) \tag{2.5}$$

$$\ddot{\delta}_2 = m_2^{-1} R_{12}(\delta_1, \dot{\delta}_1) - (m_2^{-1} + m_3^{-1}) R_{23}(\delta_2, \dot{\delta}_2)$$

If δ_1 or δ_2 is negative, the corresponding reaction R_{12} or R_{23} is equal to zero.

The solution of Eqs (2.5) must satisfy the initial conditions $\delta_1(t_0) = \delta_2(t_0) = 0$, $\dot{\delta}_1(t_0) = v_1 - v_2$, $\dot{\delta}_2(t_0) = v_2 - v_3$. The equalities $\zeta_{1,2}(t_k) = 0$, $\dot{\delta}_{1,2}(t_k) \leqslant 0$ serve as a test that the collision is completed. (In the case of plastic impact this condition is different: $\dot{\zeta}_1(t_k) = \dot{\zeta}_2(t_k) = 0$.)

The difference in the domains of variation of the variables is a characteristic feature of Eqs (2.5): the velocities $\dot{\delta}_{1,2}$ attain finite values, the coordinates $\delta_{1,2}$ and the integration interval $t_k - t_0$ are negligibly small and the accelerations $\ddot{\delta}_{1,2}$ are large. In principle, the system contains an implicit large parameter M inversely proportional to the duration $t_k - t_0$ of the impact (in mechanical systems the latter is of the order of 10^{-6}–10^{-3}) such that $\delta_{1,2} = O(M^{-1})$, $\dot{\delta}_{1,2} = O(1)$, $\ddot{\delta}_{1,2} = O(M)$. The generally accepted hypothesis that the duration of the impact can be neglected, which forms the basis for the derivation of (1.1) and (1.2), is equivalent to taking the limit as $M \to +\infty$.

To facilitate the analysis we will scale the units of length and time, setting $\delta^* = M\delta$, $\zeta^* = M\zeta$, $t^* = M(t - t_0)$. Equations (2.5) then become

$$\frac{d^2\delta_1^*}{dt^{*2}} = \frac{1}{m_2} R_{23}^*\left(\zeta_2^*, \frac{d\delta_2^*}{dt^*}\right) - \left(\frac{1}{m_1} + \frac{1}{m_2}\right) R_{12}^*\left(\zeta_1^*, \frac{d\delta_1^*}{dt^*}\right) \tag{2.6}$$

$$\frac{d^2\delta_2^*}{dt^{*2}} = \frac{1}{m_2} R_{12}^*\left(\zeta_1^*, \frac{d\delta_1^*}{dt^*}\right) - \left(\frac{1}{m_2} + \frac{1}{m_3}\right) R_{23}^*\left(\zeta_2^*, \frac{d\delta_2^*}{dt^*}\right)$$

$$R\left(\zeta^*, \frac{d\delta^*}{dt^*}\right) = \frac{1}{M} R\left(\frac{\zeta^*}{M}, \frac{d\delta^*}{dt^*}\right)$$

System (2.6) is of regular form and has unique solutions which depend continuously on the initial conditions. In particular, this means that if the initial values of δ^* are only slightly different from zero, then the finite values of the derivatives $d\delta^*/dt^*$ increase by the same infinitesimal order of magnitude. Applied to the original equations (2.5), this means that altering the initial values of the coordinates by Δ can lead to a modification of order $M\Delta$ of the velocities after the impact. In the limit as $M \to \infty$ this means that the dependence of the solution on the initial conditions is not continuous, i.e. the multiple impact problem is ill-posed.

Let us explain this ill-posedness using the previous numerical example. If the absolute values of the velocities of the first and third balls are slightly different, these balls will reach the second one at slightly different times (which is always the case in practice). Depending on which of the collisions occurs first, the result will be determined by one of the two described scenarios.

The use of the indentation method in the problem under consideration is justified in just one special case, namely, when two of the balls are in contact until the impact, their velocities being equal. Then the ambiguity will disappear and system (2.6) can be solved with the initial conditions $\delta_1^* = \delta_2^* = 0$, $\dot{\delta}_1^* = 0$, $\dot{\delta}_2^* = u_2$ (or, conversely, $\delta_2^* = 0$, $\dot{\delta}_1^* = u_1$). The indentation method can be reduced neither to independent restitution nor to consecutive "pairwise" collisions. This difference is most apparent when the bodies have different rigidity.

Example. Let $m_1 = m_2 = m_3 = 1$, $v_1 = 1$, $v_2 = v_3 = 0$ and suppose the impacts involve no dissipation. System (2.6) becomes

$$\frac{d^2\delta_1^*}{dt^{*2}} = R_{23}^*(\zeta_2^*) - 2R_{12}^*(\zeta_1^*), \quad \frac{d^2\delta_2^*}{dt^{*2}} = R_{12}^*(\zeta_1^*) - 2R_{23}^*(\zeta_2^*) \tag{2.7}$$

$$\delta_1^*(t_0) = \delta_2^*(t_0) = 0, \quad \frac{d\delta_1^*}{dt^*}(t_0) = 1, \quad \frac{d\delta_2^*}{dt^*}(t_0) = 0$$

We shall consider three cases in which the colliding bodies are made from different materials.

A. All the balls are identical. Then, according to the Hertz contact theory, $R_{12}^*(x) \equiv R_{23}^*(x) = Cx^{1.5}$. Numerical integration of system (2.7) leads to the following result: $V_1 = -0.071$, $V_2 = 0.076$, $V_3 = 0.995$.

B. The rigidity of the third body is much less than that of the first and second (a hard eraser and coins are used in experiments, the difference in rigidity reaching three orders of magnitude. Then $R_{12}^*(x) \gg R_{23}^*(x)$. The multiple impact is split into two different phases: during the first one $\delta_1^* > 0$ and R_{23}^* is negligibly small in (2.7). At the end of this phase all the momentum of the first ball is transferred to the second one and the deformation ζ_1 vanishes. During the second phase $R_{12}^* \equiv 0$. Consequently, the impulse of the second ball is transferred to the third one. As a result, we get $V_1 = V_2 = 0$, $V_3 = 1$.

We can conclude that, in the case in question, the multiple impact follows the scenario of the consecutive impact method.

C. We interchange the first and third bodies, so that $R_{12}^*(x) \ll R_{23}^*(x)$. Calculations show that in this case δ_2^* remains close to zero, so that $R_{12}^*(\zeta_1^*) \approx 2R_{23}^*(\zeta_2^*)$. As a result, $V_1 = -1/3$, $V_2 = V_3 = 2/3$, i.e. the second and third balls do not separate after the impact.

Note that in the given example the solution constructed is only realistic if $\delta_2(t_0) = 0$. If between the second and third balls there is a gap of the same order as the impact deformations (in a collision between coins these deformations do not exceed a few hundredths of a millimetre), the result obtained by the successive impact method may turn out to be closer to the truth.

4. We shall consider the given system for various admissible initial conditions in the vicinity of a multiple impact. The point of this extension is that the coordinates and the radii of the spheres contain errors, which in practice are much greater than the impact deformations. Moreover, in general we have a sequence of pairwise collisions, rather than a simultaneous collision between all three balls. Two scenarios are possible depending on which of the other two balls is first touched by the middle one, the probability of each scenario being close to a half. The intermediate case when the intervals of contact between the colliding pairs overlap is extremely improbable (the only exception being the case when two of the balls are in constant contact before the collision).

Successive pairwise collisions can be computed using Eqs (1.1) and (1.2). For the collision between the second and first balls

$$V_1 = [v_1(m_1 - \kappa_{12}m_2) + v_2m_2(1+\kappa_{12})]/(m_1+m_2) \tag{2.8}$$

$$V_2 = [v_2(m_2 - \kappa_{12}m_1) + v_1m_1(1+\kappa_{12})]/(m_1+m_2), \quad V_3 = v_3$$

and for the collision between the second and third balls

$$V_2 = [v_2(m_2 - \kappa_{23}m_3) + v_1 m_3(1+\kappa_{23})]/(m_2+m_3) \tag{2.9}$$

$$V_3 = [v_2(m_3 - \kappa_{23}m_2) + v_2 m_2(1+\kappa_{23})]/(m_2+m_3), \quad V_1 = v_1$$

The total number of pairwise collisions depends on the coefficients of restitution and the mass ratios and can be arbitrarily large and theoretically even infinite [4]. The inequalities $V_1 \leqslant V_2 \leqslant V_3$ serve as a criterion that the multiple impact has been completed. The criterion can be verified by computing $\alpha = (V_2 - V_1)/(V_3 - V_2)$. Pairwise collisions will continue until $\alpha < 0$ and cease when α becomes positive.

The change of α during a collision between the first and second balls can be described by the formula

$$\alpha^+ = \varphi_1(\alpha^-) = -\kappa_{12}\alpha^-/(1+\theta_1\alpha^-), \quad \theta_1 = m_1(1+\kappa_{12})/(m_1+m_2) \tag{2.10}$$

and for a collision between the second and third balls by the formula

$$\alpha^+ = \varphi_2(\alpha^-) = -(\alpha^- + \theta_2)/\kappa_{23}, \quad \theta_2 = m_3(1+\kappa_{23})/(m_2+m_3) \tag{2.11}$$

Let the value before the impact be α_0. The chain

$$\alpha_1 = \varphi_1(\alpha_0), \quad \alpha_2 = \varphi_2(\alpha_1), \quad \alpha_3 = \varphi_1(\alpha_2), \ \ldots \tag{2.12}$$

corresponds to the first possible sequence of collisions, the total number of collisions k_1 being determined by the condition $\alpha_{k_1} > 0$.

The second sequence is described by the relations

$$\alpha_1 = \varphi_2(\alpha_0), \quad \alpha_2 = \varphi_1(\alpha_1), \quad \alpha_3 = \varphi_2(\alpha_2), \ \ldots \tag{2.13}$$

The number of collisions k_2 may be different from k_1.

As a rule, (2.12) and (2.13) lead to different results, which indicates that the system is stochastic. Some exceptions are also possible when the results are the same. Here an appropriate analogy is with the collision between a material point and the vertex of a dihedral angle mentioned in Section 1.

We set

$$\mathbf{q} = (x_1, x_2, x_3), \quad f_1 = x_2 - x_1 - \rho_1 - \rho_2 \geqslant 0, \quad f_2 = x_3 - x_2 - \rho_2 - \rho_3 \geqslant 0$$

The system has a kinetic energy

$$T = \tfrac{1}{2}\sum_{j=1}^{3} m_j \dot{x}_j^2 = \tfrac{1}{2}\dot{\mathbf{q}}\mathbf{A}\dot{\mathbf{q}}^{\mathrm{T}}, \quad \mathbf{A} = \operatorname{diag}\{m_1, m_2, m_3\}$$

The angle between the planes $f_1 = 0$ and $f_2 = 0$ in the metric (1.4) can be computed from the formula [4]

$$\begin{gathered} \cos\beta = -(\mathbf{e}_1, \mathbf{A}^{-1}\mathbf{e}_2)(\mathbf{e}_1, \mathbf{A}^{-1}\mathbf{e}_1)^{-1/2}(\mathbf{e}_2, \mathbf{A}^{-1}\mathbf{e}_2)^{-1/2} \\ \mathbf{e}_1 = \operatorname{grad} f_1 = (-1, 1, 0), \quad \mathbf{e}_2 = \operatorname{grad} f_2 = (0, -1, 1) \end{gathered} \tag{2.14}$$

It follows that $\cos\beta = (m_1 m_3)^{1/2}(m_1 + m_2)^{-1/2}(m_3 + m_2)^{-1/2}$. We note that β is an acute angle for any mass ratios.

Example. Let $m_1 = m_3$ and $\kappa_{12} = \kappa_{23} = \kappa$. Then $\theta_1 = \theta_2 = \theta$ and $\cos\beta = \theta/(1+\kappa)$.
The equation

$$\varphi_1 \circ \varphi_2 = \varphi_2 \circ \varphi_1$$

has a unique solution $\kappa = 1$, $\theta = \sqrt{2}$. Moreover $\varphi_1 \circ \varphi_2(\alpha_0) < 0$ for all $\alpha_0 < 0$ and $\varphi_1 \circ \varphi_2 \circ \varphi_1 \circ \varphi_2(\alpha_0) > 0$. Consequently, the multiple impact can be reduced to four pairwise collisions.

Next, the equation

$$\varphi_1 \circ \varphi_2 \circ \varphi_1 = \varphi_2 \circ \varphi_1 \circ \varphi_2$$

has a unique solution $\kappa = 1$, $\theta = \sqrt{3}$ and the impact can be reduced to three pairwise collisions. By analogy, one can form chains of arbitrary length m ($m = 5, 6, \ldots$). It turns out that in all cases regularity can be achieved only if $\kappa = 1$, $\theta = \theta_m = 2\cos(\pi/m)$. From the dynamical point of view, the system in question turns out to be equivalent to an absolutely rigid collision between a particle and a plane angle $\beta = \arccos(\theta/2)$, which is regular only when $\beta = \pi/m$ ($m = 2, 3, \ldots$) [4, 5].

In each of the discovered quasiregular cases the solution of the multiple impact problem has the same form

$$V_j = -v_j + 2(m_1v_1 + m_2v_2 + m_3v_3)/(m_1 + m_2 + m_3) \quad (j = 1,2,3) \tag{2.15}$$

Apart from this, there is also the case of a quasiplastic impact, in which the sequence of pairwise collisions is theoretically infinite. The condition for it to occur, obtained in [4], is

$$\frac{m_1}{m_1 + m_2} > \frac{4\sqrt{\kappa}}{(1+\kappa)^2} \tag{2.16}$$

and the result has the form

$$V_1 = V_2 = V_3 = (m_1v_1 + m_2v_2 + m_3v_3)/(m_1 + m_2 + m_3)$$

Apart from the above-mentioned cases it is also possible that the two multiple impact scenarios are the same for different values of α characterizing the initial conditions. All the cases listed are instances of multiple impact of quasiregular type.

Let us summarize the results. If the multiple impact problem admits of error in the initial conditions, the result must be determined by the statistical ensemble method, i.e. various versions of successive collisions must be considered. As a result, the problem will usually have two different equally probable solutions (the stochastic type), which may, however, be identical for some parameter values (the quasiregular type).

3. THREE-DIMENSIONAL COLLISION BETWEEN THREE BALLS

We will now consider the general three-dimensional case of three colliding balls with smooth surfaces. The reactions R_{21} and R_{23} are vectors parallel to the straight lines G_2G_1 and G_2G_3 connecting the centres of the balls (Fig. 2a). Because they do not give rise to rotational momenta, the impact can be reduced to a modification of the velocities $\dot{\mathbf{r}}_j$ ($j = 1, 2, 3$) of the centres of the balls.

By analogy with (2.4), one can set up the following equations for the impact

$$m_1[\dot{\mathbf{r}}_1(t) - \dot{\mathbf{r}}_1(t_0)] = \mathbf{I}_1, \quad m_2[\dot{\mathbf{r}}_2(t) - \dot{\mathbf{r}}_2(t_0)] = -\mathbf{I}_1 - \mathbf{I}_2 \tag{3.1}$$

$$m_3[\dot{\mathbf{r}}_3(t) - \dot{\mathbf{r}}_3(t_0)] = \mathbf{I}_2$$

$$\mathbf{I}_1 = R_{21}\mathbf{n}_1, \quad \mathbf{I}_2 = R_{23}\mathbf{n}_2, \quad \mathbf{n}_1 = G_2G_1/|G_1G_2|, \quad \mathbf{n}_2 = G_2G_3/|G_3G_2|$$

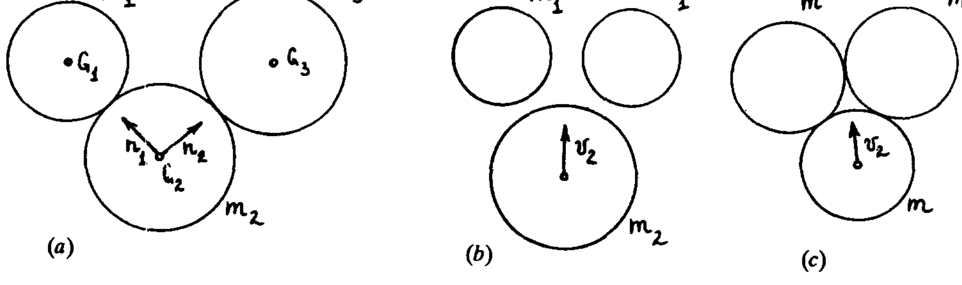


Fig. 2.

Contact deformations are defined by

$$\delta_1 = \rho_1 + \rho_2 - |G_1G_2|, \quad \delta_2 = \rho_2 + \rho_3 - |G_2G_3|$$

Their time derivatives are

$$\dot{\delta}_1 = -d/dt|\mathbf{r}_2 - \mathbf{r}_1| = (\dot{\mathbf{r}}_2 - \dot{\mathbf{r}}_1, \mathbf{n}_1), \quad \dot{\delta}_2 = (\dot{\mathbf{r}}_2 - \dot{\mathbf{r}}_3, \mathbf{n}_2) \tag{3.2}$$

Differentiating (3.1) and taking (3.2) into account, we obtain equations for the impact of the form

$$\begin{aligned}
\ddot{\delta}_1 &= -m_2^{-1}(\mathbf{n}_1, \mathbf{n}_2)R_{23}(\delta_2, \dot{\delta}_2) - (m_1^{-1} + m_2^{-1})R_{21}(\delta_1, \dot{\delta}_1) \\
\ddot{\delta}_2 &= -m_2^{-1}(\mathbf{n}_1, \mathbf{n}_2)R_{21}(\delta_1, \dot{\delta}_1) - (m_2^{-1} + m_3^{-1})R_{23}(\delta_2, \dot{\delta}_2) \\
\delta_1(t_0) &= \delta_2(t_0) = 0, \quad \dot{\delta}_1(t_0) = \dot{\delta}_1^0, \quad \dot{\delta}_2(t_0) = \dot{\delta}_2^0
\end{aligned} \tag{3.3}$$

In particular, for a collinear collision $\mathbf{n}_1 = -\mathbf{n}_2$, and we obtain (2.5).

Equations (3.3) can be separated if $\mathbf{n}_1$ and $\mathbf{n}_2$ are orthogonal

$$(\mathbf{n}_1, \mathbf{n}_2) = 0 \tag{3.4}$$

In this case the multiple impact can be reduced to the sum of two independent pairwise collisions and the solution of the problem reads

$$\begin{aligned}
\mathbf{V}_1 &= \mathbf{v}_1 + (1 + \kappa_{12})(\mathbf{v}_2 - \mathbf{v}_1, \mathbf{n}_1)m_2(m_1 + m_2)^{-1}\mathbf{n}_1 \\
\mathbf{V}_3 &= \mathbf{v}_3 + (1 + \kappa_{23})(\mathbf{v}_2 - \mathbf{v}_3, \mathbf{n}_2)m_2(m_3 + m_2)^{-1}\mathbf{n}_2 \\
\mathbf{V}_2 &= \mathbf{v}_2 + \theta_1(\mathbf{v}_1 - \mathbf{v}_2, \mathbf{n}_1)\mathbf{n}_1 + \theta_2(\mathbf{v}_3 - \mathbf{v}_2, \mathbf{n}_1)\mathbf{n}_2
\end{aligned} \tag{3.5}$$

where the parameters $\theta_{1,2}$ are defined in (2.10).

If condition (3.4) is violated, an exact solution of Eqs (3.3) is impossible, in general. Random errors in the initial conditions will result in the spheres hitting one another at different times. The result may differ significantly depending on which collision occurs first. As in Section 2, one can draw an analogy with a mass point colliding with the vertex of a dihedral angle. As has been observed above, an absolutely rigid impact is well defined if the angle is an integral part of π, i.e. $\beta_m = \pi/m$ ($m \in N$).

We introduce a Cartesian system of coordinates $OXYZ$ and we denote by (x_j, y_j, z_j) the coordinates of the centres G_j ($j = 1, 2, 3$) of the balls. The kinetic energy can be expressed by the formula

$$T = \tfrac{1}{2}\sum_{j=1}^{3} m_j(\dot{x}_j^2 + \dot{y}_j^2 + \dot{z}_j^2) = \tfrac{1}{2}\dot{\mathbf{q}}\mathbf{A}\dot{\mathbf{q}}^{\mathrm{T}}$$

$$\mathbf{A} = \mathrm{diag}\{m_1, m_2, m_3, m_1 m_2, m_3, m_1, m_2, m_3\}, \quad \mathbf{q} = (x_1, x_2, x_3, y_1, y_2, y_3, z_1, z_2, z_3)$$

The unilateral constraints imposed on the system can be given by the inequalities

$$\begin{aligned}
f_1 &= (x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2 - (\rho_1 + \rho_2)^2 \geqslant 0 \\
f_2 &= (x_3 - x_2)^2 + (y_3 - y_2)^2 + (z_3 - z_2)^2 - (\rho_3 + \rho_2)^2 \geqslant 0
\end{aligned} \tag{3.6}$$

We define the normal vectors $\mathbf{n}_{1,2}$ to the surfaces $f_{1,2} = 0$ by (1.6)

$$\mathbf{n}_1 = \left(\frac{1}{m_1}(x_2 - x_1), \frac{1}{m_2}(x_1 - x_2), 0, \frac{1}{m_1}(y_2 - y_1), \frac{1}{m_2}(y_1 - y_2), 0, \frac{1}{m_1}(z_2 - z_1), \frac{1}{m_2}(z_1 - z_2), 0\right)$$

$$\mathbf{n}_2 = \left(0, \frac{1}{m_2}(x_2 - x_3), \frac{1}{m_3}(x_3 - x_2), 0, \frac{1}{m_2}(y_2 - y_3), \frac{1}{m_3}(y_3 - y_2), 0, \frac{1}{m_2}(z_2 - z_3), \frac{1}{m_3}(z_3 - z_2)\right)$$

We obtain the following expression for the angle between the surfaces (3.6)

$$\cos\beta = -(1+m_2/m_1)^{-1/2}(1+m_2/m_3)^{1/2}\cos\angle G_1G_2G_3 \tag{3.7}$$

Condition (3.4) is therefore equivalent to the orthogonality of the surfaces (3.6) in the Jacobi metric.

Example. (Bernoulli's problem.) A ball of mass m_2 hits two identical stationary symmetrically placed balls (Fig. 2b). Dissipation of energy is neglected. A solution of this problem can be obtained if the laws of conservation of energy and momentum are supplemented by the condition that symmetry should be preserved after the impact [1]. However, experiments indicate that the result will not reflect the behaviour of a real system. If the angle $G_1G_2G_3$ formed by the centres at the time of collision is acute, any slight initial asymmetry of the system may lead to a substantial difference between the velocities of the first and third balls after the impact. If $m_2 < m_1$, the multiple impact can be reduced to the ordinary collision of a moving ball with one of the two stationary balls without contact with the other stationary ball. In general, it can be represented as a sequence of "pairwise" collisions, the order of collisions being random. The length of the sequence depends on β, which can be computed from the formula

$$\cos\beta = -\frac{m_1}{m_1+m_2}\cos\angle G_1G_2G_3$$

since $m_1 = m_3$. The preservation of symmetry after the impact, which Bernoulli considered as a postulate, can take place in the given problem only in quasiregular cases when β is an integral part of a right angle, i.e.

$$\cos\angle G_1G_2G_3 = -(1+m_2/m_1)\cos\pi/k \quad (k=2,3,\ldots) \tag{3.8}$$

The criterion (3.8) is satisfied independently of the mass ratio if the angle formed by the centres of the balls is a right angle ($k = 2$). If $m_2 \geqslant m_1$, then there are no other solutions because the absolute value of the right-hand side is greater than unity. When $m_2 < m_1$, Eq. (3.8) has several solutions, the number of which increases as m_1/m_2 increases.

In particular, suppose that all three balls are identical, the first and third being in contact before the impact (Fig. 2c). Then the centres of the balls form an equilateral triangle at the time of impact (ignoring random errors). From (3.7) we get $\cos\beta = -0.25$, which implies that $\beta \approx 0.58\pi$. It follows that (3.8) is not satisfied and the system is of stochastic type. Straightforward computations demonstrate that a collision with the first ball followed by a collision with the third one leads to the following result

$$\mathbf{V} = \frac{\sqrt{3}}{2}|\mathbf{v}_2|n_1,\quad \mathbf{V}_2 = \frac{\sqrt{3}}{12}|\mathbf{v}_2|(\mathbf{n}_2-2\mathbf{n}_1),\quad \mathbf{V}_3 = \frac{\sqrt{3}}{4}|\mathbf{v}_2|\mathbf{n}_2 \tag{3.9}$$

The opposite order of pairwise collisions, namely, when the second ball hits the third one first, followed by a collision with the first ball, leads to a different result

$$\mathbf{V}_1 = \frac{\sqrt{3}}{4}|\mathbf{v}_2|n_1,\quad \mathbf{V}_2 = \frac{\sqrt{3}}{12}|\mathbf{v}_2|(\mathbf{n}_1-2\mathbf{n}_2),\quad \mathbf{V}_3 = \frac{\sqrt{3}}{2}|\mathbf{v}_2|\mathbf{n}_2 \tag{3.10}$$

Depending on random errors, the problem has two equally probable solutions (3.9) and (3.10). In both cases one of the balls that were initially stationary will attain a velocity double that of the other one after the impact.

The stochastic nature of the problem will also be retained if we assume that the collisions are absolutely non-elastic, which was studied in [2]. Straightforward computations indicate that the moving ball will be deflected from the axis of symmetry after consecutive collisions with symmetrically placed balls.

Apart from the aforementioned cases, in which the correct solution of the problem of the three-dimensional impact of three balls can be obtained, a collision in which two balls are in contact and are stationary with respect to one another is also of quasiregular type. The solution can be found by the indentation method.

4. THE COLLISION OF THREE BODIES OF ARBITRARY SHAPE

We shall now consider a collision between three rigid bodies of arbitrary shape with smooth surfaces. As before, we neglect the dimensions of the domain of contact and assume that the first and second bodies have one common point C_1 and the second and third bodies have a common point C_2. Denoting by G_k ($k = 1, 2, 3$) the centres of mass of the bodies, by $\mathbf{J}_k$ their central inertia tensors, and by $\mathbf{W}_k$ their angular velocities, we can represent the equations of the impact in the following form [6], similar to (3.1)

$$m_1[\dot{\mathbf{r}}_1(t)-\dot{\mathbf{r}}_1(t_0)] = \mathbf{I}_1,\quad m_2[\dot{\mathbf{r}}_2(t)-\dot{\mathbf{r}}_2(t_0)] = -\mathbf{I}_1-\mathbf{I}_2, \tag{4.1}$$

$$m_3[\dot{\mathbf{r}}_3(t)-\dot{\mathbf{r}}_3(t_0)]=\mathbf{I}_2,\quad \mathbf{J}_1[\mathbf{W}_1(t)-\mathbf{W}_1(t_0)]=G_1C_1\times\mathbf{I}_1$$

$$\mathbf{J}_3[\mathbf{W}_3(t)-\mathbf{W}_3(t_0)]=G_3C_2\times\mathbf{I}_2,\quad \mathbf{J}_2[\mathbf{W}_2(t)-\mathbf{W}_2(t_0)]=-G_2C_1\times\mathbf{I}_1-G_2C_2\times\mathbf{I}_2$$

The normal deformations are defined by

$$\delta_1=\left(\mathbf{r}_{c_1}^{(1)}-\mathbf{r}_{c_1}^{(2)},\mathbf{n}_1\right),\quad \delta_2=\left(\mathbf{r}_{c_2}^{(3)}-\mathbf{r}_{c_2}^{(2)},\mathbf{n}_2\right) \tag{4.2}$$

Here $\mathbf{r}_c^{(j)}$ is the position vector of a point C on the non-deformed surface of the jth body and $\mathbf{n}_{1,2}$ are the unit vectors normal to the contact surface. Then the deformation rate can be computed from the formulae

$$\dot{\delta}_1=\left(\dot{\mathbf{r}}_{c_1}^{(1)}-\dot{\mathbf{r}}_{c_1}^{(2)},\mathbf{n}_1\right),\quad \dot{\delta}_2=\left(\dot{\mathbf{r}}_{c_2}^{(3)}-\dot{\mathbf{r}}_{c_2}^{(2)},\mathbf{n}_2\right) \tag{4.3}$$

We express the relative velocities by the Euler formulae $\dot{\mathbf{r}}_c^{(j)}=\dot{\mathbf{r}}_j+\mathbf{W}_j\times G_jC$ and compute the second derivatives of the deformations by differentiating (4.3). As a result, we obtain

$$\ddot{\delta}_1=\left(\frac{1}{m_1}\mathbf{R}_1+\frac{1}{m_2}(\mathbf{R}_1+\mathbf{R}_2)+\mathbf{J}_1^{-1}(G_1C_1\times\mathbf{R}_1)\times G_1C_1+\mathbf{J}_2^{-1}(G_2C_1\times\mathbf{R}_1)\times G_2C_1+\right. \tag{4.4}$$

$$\left.+\mathbf{J}_2^{-1}(G_2C_2\times\mathbf{R}_2)\times G_2C_1,\mathbf{n}_1\right.$$

$$\ddot{\delta}_2=\left(\frac{1}{m_3}\mathbf{R}_2+\frac{1}{m_2}(\mathbf{R}_1+\mathbf{R}_2)+\mathbf{J}_3^{-1}(G_3C_2\times\mathbf{R}_2)\times G_3C_2+\mathbf{J}_2^{-1}(G_2C_1\times\mathbf{R}_1)\times G_2C_2+\right.$$

$$\left.+\ \mathbf{J}_2^{-1}(G_2C_2\times\mathbf{R}_2)\times G_2C_2,\mathbf{n}_2)\right)$$

In (4.4) each of the reactions $\mathbf{R}_{1,2}$ is parallel to the corresponding normal vector $\mathbf{n}_{1,2}$ and depends only on the corresponding deformation and its rate of change. The condition that the multiple impact problem should admit of a correct solution is that the right-hand side of the first equation should be independent of $\mathbf{R}_1$ and that of the second should be independent of $\mathbf{R}_2$, and can be expressed by a single equality

$$(\mathbf{n}_1,\mathbf{n}_2)+m_2(\mathbf{J}_2^{-1}(G_2C_2\times\mathbf{n}_2),\ G_2C_1\times\mathbf{n}_1)=0 \tag{4.5}$$

It is interesting that condition (4.5) relates only the dynamical characteristics of the second body to the positions of the contact points on its surface during the impact, and it is independent of the properties of the other two bodies. If at least one of the vectors G_2C_1 or G_2C_2 is orthogonal to the surface of the body, this condition can be reduced to (3.4).

Equality (3.4) can also be obtained geometrically as a condition ensuring that the impact surfaces are orthogonal in the Jacobi metric. The argument is similar to the discussion in the previous sections but more complicated because the dimension of the configuration space is equal to 18 in the case under consideration. We shall therefore restrict ourselves to the regularity condition (4.5), bearing in mind that there are also quasiregular cases (for example, an absolutely rigid impact with $\beta=\pi/k$).

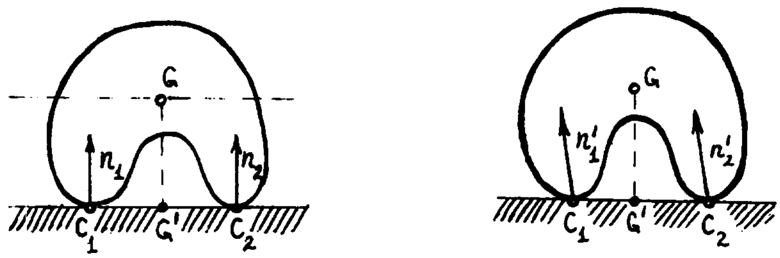
Example. Suppose that a body shaped like a horseshoe hits a barrier at two points simultaneously (plane impact, Fig. 3a). In this case the horseshoe plays the role of the central body and the massive support serves as the first and third bodies. We have $\mathbf{n}_1=\mathbf{n}_2$ and $m_2\mathbf{J}_2^{-1}=\rho^{-2}\mathbf{E}_2$ (ρ is the radius of inertia and $\mathbf{E}_2$ is the identity matrix). The equality (4.5) takes the form

$$|G'C_1|*|G'C_2|=\rho^2 \tag{4.6}$$

where G' is the projection of the centre of mass onto the support.

This relationship means that if the body is attached at one of the contact points C_1 or C_2, the other point will lie on the line of action of the impact impulse applied to the impact centre. Therefore the impact reaction at either of the contact points does not give rise to a load at the other point.

The above example can be extended to the case when the surfaces of colliding bodies are rough and the impact forces have tangential components. If the angle of incidence is large enough, sliding does not stop during the impact,



(a) (b)

Fig. 3.

and the direction of the reaction remains unchanged and parallel to the vectors $\mathbf{n}'_1 = \mathbf{n}'_2 = (-\sin\alpha, \cos\alpha)$ with tg $\alpha = \mu$, where μ is the coefficient of dynamic friction (Fig. 3b). In this case (4.5) will take the following form after reduction

$$(1+\mu^2)\rho^2 = |G'C_1| * |G'C_2| - \mu^2|GG'|^2 + \mu|GG'|\left(|G'C_1| - |G'C_2|\right)$$

In the general case of multiple impact between rough bodies the impact impulses can be correctly determined only for those systems in which the direction of relative sliding remains unchanged during the impact. The examples of such systems are probably exhausted by the case of plane parallel motion.

5. COLLISION BETWEEN MANY BODIES

The results obtained above can be extended to the case of a simultaneous collision between more than three rigid bodies. To obtain the regularity conditions one must consider all possible triples of bodies in the system forming two impact pairs and write down equalities of type (4.5) for each of them. It turns out that the total number of colliding bodies for which the orthogonality conditions can be satisfied is unlimited; however, no one body must hit more than six other bodies. The restriction can be explained using the geometric representation of a multiple impact: each body has six degrees of freedom and there can be at most six pairwise-orthogonal (in the Jacobi metric) vectors (impact impulses) in a six-dimensional space. Besides, a very special shape of the body and a special selection of the points of contact with other bodies are required to attain this maximum value. In particular, only three directions perpendicular to one another are possible for a ball.

The following example gives some idea of more complex cases of pairwise orthogonality.

Example. Again, we consider a horseshoe (Fig. 3a), for which the orthogonality condition (4.6) is satisfied. Along with two impacts orthogonal to one another at the points C_1 and C_2, there is a third impact orthogonal to each of the two. Its line of action passes through the centre of mass G and is parallel to the line C_1C_2. Indeed, such an impulse causes the horseshoe to move forward, the points of contact with the obstacle moving along in the same direction. However, such an impact is possible only when the straight line through G parallel to C_1C_2 intersects the boundary of the horseshoe at a right angle.

For a plane body there are three degrees of freedom and this example gives the maximum number of orthogonal impacts.

If the orthogonality condition is violated, the statistical ensemble method should be used. The resulting solution will be qualitatively more complex than in the cases considered above: the number of possible versions of consecutive collisions can be as large as desired, each having a different probability of realization.

This is so because the geometry of a trihedral angle (the more so of a polyhedral angle) is more complex compared with a dihedral one because its faces have different apparent angular dimensions, which also depend on the position of the observer. Because of this, random deviations from the trajectory leading to the vertex result in different faces of the angle being hit with different probabilities. For a quantitative estimate of various possibilities one must define a probability measure in the phase space of the system describing the random errors in the coordinates and velocities. The neighbourhood of the unperturbed trajectory (leading to the tip of the polyhedral angle) is divided by manifolds of codimension one into parts corresponding to the various faces being hit by the representative point. Each of these parts is, in turn, subdivided into domains corresponding to different versions of the second impact, and so on. Computing the measure of each of the subdomains, we obtain a solution of the multiple impact problem as a random variable.

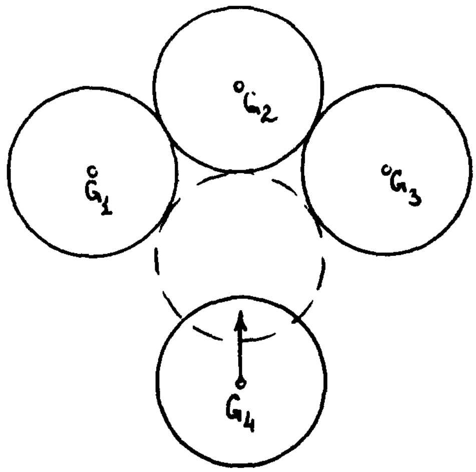


Fig. 4.

Examples. 1. We consider the system in Fig. 2(c) again, but for different initial conditions: we shall assume that the velocities of all three balls have unit absolute value and are directed towards the common centre of symmetry. The system has three collision pairs taking place one after another due to random errors. An absolutely rigid collision between the first and second balls and the between the first and third balls will be followed by one more collision between the second and third balls. Then the balls will move away with velocities

$$\mathbf{V}_1 = 5\frac{\sqrt{3}}{6}\mathbf{n}_1 - \frac{\sqrt{3}}{6}\mathbf{n}_2, \quad \mathbf{V}_2 = \frac{\sqrt{3}}{12}(\mathbf{n}_2 - 8\mathbf{n}_1), \quad \mathbf{V}_3 = \frac{\sqrt{3}}{12}(\mathbf{n}_2 - 2\mathbf{n}_1) \tag{5.1}$$

If the collisions follow the sequence (2–3), (1–3), (1–2), it suffices to interchange the vectors $\mathbf{n}_1$ and $\mathbf{n}_2$ in (5.1) to obtain the solution. By analogy, one can also construct the solutions in the four remaining possible cases. Therefore, there are six versions of multiple impact in this example, the probability of each being close to one sixth.

2. We will consider a version of the Bernoulli problem, namely, the collision between a moving ball and a system of three stationary balls identical to the moving one and having two points of contact (Fig. 4). We adopt the following model of the errors: the point G_2 lies precisely on the line of motion of the fourth ball, $\angle G_4C_2G_1 = \pi/3 + \Delta$, and $\angle G_4G_2G_3 = \pi/3 + \Delta_2$, where $\Delta_{1,2}$ are identically distributed random quantities. If $\Delta_1 > 0$ and $\Delta_2 > 0$, the collision can be reduced to one pairwise impact between the fourth and second balls (the probability of this event being close to 0.25). If $\Delta_1 < 0$ and $\Delta_1 < \Delta_2$, the fourth ball will initially hit the first one, then the second one, and finally the third one. As a result, the absolute values of the velocities of the balls will be equal to 1/2, 3/4, 3/8 and $\sqrt{(3/8)}$ of the initial velocity of the fourth ball. Another possibility, when $\Delta_2 < 0$ and $\Delta_2 < \Delta_1$, leads to the same result with the first and third balls interchanged. By symmetry, each of these two events has a probability close to 0.375.

6. COLLISION OF CONSTRAINED BODIES

This problem is the most involved one in impact theory. This is because it is mixed up with the problem of the impulse acting on a system with constraints, the solution of which is well known [10]. This solution is based on the assumption that the constraints are absolutely rigid, which is why they can preserve the configuration of the system under impulsive forces. A ballistic pendulum serves as an example in which this approach is justified [10]: the impulse arises when the projectile hits a reservoir with soil, which is part of the pendulum. Another example is a billiard ball lying on a table and hit by a cue [11].

However, an extension of this method to the problem of collision between constrained bodies may lead to a false result.

Example. A billiard ball next to a fence can be regarded as a system with a constraint. Absolute rigidity of the constraint would mean that the ball remains stationary when hit perpendicular to the fence. This can in fact be achieved if a ball made of bone is hit by a rubber one. But if the balls are identical, the stationary ball will move away from the fence following the impact. To describe this phenomenon one can use (2.6) setting $m_1 = m_2 \ll m_3$ and $R_{23}^*(x) = 2\sqrt{(2R_{12}^*)(x)}$, which corresponds to the agreement between the Hertz theory and the limiting case of a fence that is absolutely rigid (as compared to the ball). By numerical integration we find that $V_1 = -0.96$ and $V_2 = -0.28$.

Remarks. 1. The problem of collision between billiard balls was solved in [12] under the assumption that the contact between the ball and the table is absolutely rigid. As follows from the above example, this assumption is justified only in the case when the incident ball is made from a much less rigid material than the other one.

2. In the example considered the constraint is unilateral. However, the argument showing that it is necessary to take its rigidity into account remains valid in the case of a bilateral constraint also. Thus, if one of the points of the stationary ball were attached to the fence, making it impossible for it to separate from the latter, this would in no way affect the velocity of the first ball as it moves away. As follows from the above results, it will loose some of its kinetic energy, which means that the energy will be transformed into vibrations of the "stationary" system consisting of the other ball and the fence.

We will now consider the problem of a collision between a physical pendulum and a fixed wall. We take the plane through the point of contact that is perpendicular to the axis of rotation and consider the resulting cross-section (Fig. 5). We shall assume the wall to be smooth, so that the impact reaction is orthogonal to it.

The pendulum can be considered as a central body colliding simultaneously with two others, namely, with the obstacle and the axis of suspension. In this problem the hinge bearing of the axis is usually assumed to be absolutely rigid. Then the impulse at C can be determined from Newton's hypothesis. Because there is only one degree of freedom, namely, the rotational one, it follows that $\mathbf{I}$ is uniquely defined by (1.1). Then one can compute the reactions at the points where the pendulum is attached [10].

This approach to the solution appears faultless at first sight; however, a flaw can be detected in it. Indeed, the assumption of ideal attachment to the axis of rotation is only justified in one case when the material from which the obstacle is made is much less rigid than the pendulum and the axis of suspension. The duration of impact is then long enough to make it possible to neglect the compliance of the support. But if the pendulum and the wall are made from materials whose rigidity is of the same order, then vibrations occur at the points of attachment causing dissipation of some kinetic energy. The greater the impact loads necessary to keep the axis of suspension of the pendulum fixed, the higher the dissipation.

To obtain quantitative estimates we will restrict ourselves to the plane case (Fig. 5). Remaining within the framework of the absolutely rigid body hypothesis, we will consider the following model of elastic suspension.

We shall assume that the stationary support O^* and the fixed point O of the body are connected by a rigid spring preventing their separation. The reaction of the support depends on the displacement of O and is given by

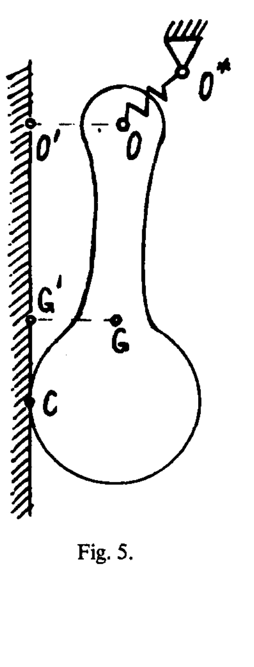


Fig. 5.

$$\mathbf{R}_o = -K(q)\mathbf{q}, \quad \mathbf{q} = O^*O \tag{6.1}$$

In (6.1) the absolute value of the coefficient of rigidity K is assumed to be large.

Equations (4.1) take the form

$$\begin{gathered} m_2[\dot{\mathbf{r}}_2(t) - \dot{\mathbf{r}}_2(t_0)] = -\mathbf{I}_1 - \mathbf{I}_0, \quad \mathbf{J}_2[\mathbf{W}_2(t) - \mathbf{W}_2(t_0)] = -G_2C \times \mathbf{I}_1 - G_2O \times \mathbf{I}_0 \\ \mathbf{I}_0 = -\int_{t_0}^{t} \mathbf{R}_0 dt, \quad \mathbf{I}_1 = \int_{t_0}^{t} \mathbf{R}_1 dt, \quad \mathbf{R}_1 = R_1(\delta_1, \dot{\delta}_1)\mathbf{n}_1 \end{gathered} \tag{6.2}$$

where the reaction $\mathbf{R}_0$ can be computed from (6.1) and the deformation δ_1 from (4.2).

The velocity of O is given by

$$\mathbf{V}_o = \dot{\mathbf{q}} = \dot{\mathbf{r}}_2(t) + \mathbf{W}(t) \times G_2O \tag{6.3}$$

Combining (6.1)–(6.3), we obtain impact equations of the form

$$\begin{gathered} \ddot{\mathbf{q}} = m_2^{-1}(\mathbf{R}_1 - K\mathbf{q}) + \mathbf{J}_2^{-1}(G_2C \times \mathbf{R}_1 - G_2O \times K\mathbf{q}) \times G_2O \\ \ddot{\delta} = m_2^{-1}(-\mathbf{R}_1 + K\mathbf{q}, \mathbf{n}_1) + \left(\mathbf{J}_2^{-1}(-G_2C \times \mathbf{R}_1 + G_2O \times K\mathbf{q}), G_2C \times \mathbf{n}_1\right) \\ \mathbf{q}(t_0) = \dot{\mathbf{q}}(t_0) = 0, \quad \delta_1(t_0) = 0, \quad \dot{\delta}_1(t_0) = \left(OC \times \mathbf{W}(t_0), \mathbf{n}_1\right) \end{gathered} \tag{6.4}$$

Formulae (6.4) can also be used to describe a three-dimensional impact between a rigid body with a stationary point and an obstacle.

In the general case the variables $\mathbf{q}$, δ_1 in (6.4) are related to one another, which makes it necessary to prescribe K as a function of $R_1(\delta_1, \dot{\delta}_1)$ to solve the impact problem. For the system under consideration only the fourth initial condition can vary, the first three conditions being fixed. Given that, the solution of (6.4) will depend continuously on the initial conditions. Nevertheless, the solution depends strongly on the ratio of the rigidities at the points where the pendulum is in contact with the support and with the axis of suspension. Therefore the problem is of the quasiregular type described in the preamble.

The only, though welcome exception is when the equality

$$m_2^{-1}(q, \mathbf{n}_1) + (\mathbf{J}_2^{-1}(G_2O \times \mathbf{q}, \ G_2C \times \mathbf{n}_1) = 0$$

is satisfied for any vector $\mathbf{q}$. In the plane case under consideration it takes the form

$$\rho^2(q, \mathbf{n}_1) + \left(G_2O \times \mathbf{q}, \ G_2C \times \mathbf{n}_1\right) = 0 \tag{6.5}$$

where ρ is the radius of inertia of the pendulum. Setting $\mathbf{q} \perp \mathbf{n}_1$ and then $\mathbf{q} = \mathbf{n}_1$ in (6.5), we arrive at the following regularity conditions

$$G_2O \perp \mathbf{n}_1, \quad \rho^2 = (O'G', G'C) \tag{6.6}$$

Here O' and G' are the projections of the corresponding points onto the obstacle (Fig. 5). Relations (6.6) mean that the reaction of the wall does not produce a load at the point of suspension, i.e. its line of action passes through the centre of impact (the latter lies on the OG_2 axis, which is parallel to the wall at the time of impact).

Example. Consider an impact between an obstacle and a pendulum consisting of a weightless rod of length L and two weights of mass m_1 and m_2, the first of which is attached at the end of the rod and the other at a distance $l \cdot L$ from the point of suspension (Fig. 6). Computations lead to the following expressions for the radius of inertia and the position of the centre of mass of the system

$$OG = \frac{m_1 L + m_2 l}{m_1 + m_2}, \quad \rho_2 = \frac{m_1 L^2 + m_2 l^2}{m_1 + m_2} \tag{6.7}$$

The second condition in (6.6), taking (6.7) into account, becomes

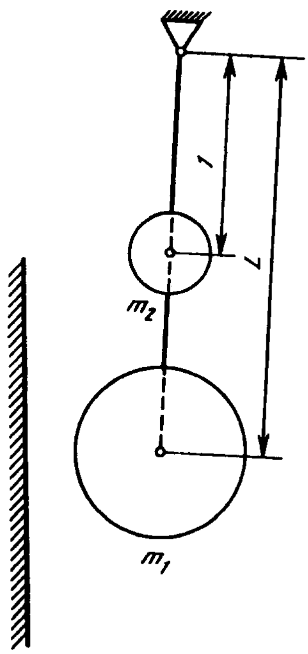


Fig. 6.

$$\left(m_1L^2+m_2l^2\right)(m_1+m_2)=m_2(L-l)(m_1L+m_2l) \tag{6.8}$$

The following experiment was carried out: all parameters of the system except l were kept constant ($m_2 = 4m_1$). It turned out that the coefficient of restitution of the relative velocity is also altered when the pendulum hits a vertical wall (the axis being vertical). The largest value of this coefficient corresponds to the root $l = L/6$ of Eq. (6.8). It is about twice as large as the value corresponding to the extreme position $l = L$.

7. CONCLUSIONS

Above we considered a number of problems involving multiple impact. The analysis carried out shows that the methods of solving these problems, as well as their solvability, are determined to a large extent by the configuration of the system at the time of impact. The following classification of possible cases can be proposed

1. In the regular case the impact pairs act independently of one another. The multiple impact problem splits into several simpler problems involving a collision between two bodies. Each of these problems can be solved within the framework of the given discrete system using the standard hypothesis concerning the coefficient of restitution without resorting to the theory of elasticity.

Quite rigid conditions, expressing the orthogonality of the impact impulses in the Jacobi metric, are required for this case to be realized (see [4]). The mechanical meaning of the orthogonality conditions (3.4), (4.5), (6.6), (6.8), etc. is that no one impact pair gives rise to a load acting on another pair.

2. The second, more frequently used, type of problems admitting of a correct solution can be referred to as quasiregular. It is characterized by the fact that the orthogonality conditions are violated, which makes it impossible to solve the problem within the framework of discrete system dynamics. To solve the problem, one can, for example, use some model of the theory of elasticity reflecting the physical properties of the colliding bodies. The result depends continuously on the initial conditions. It differs from the regular case in that the result will change substantially if the physical properties of the colliding bodies (for example, the rigidity) are altered. The problem of a collision between a physical pendulum and an obstacle as well as a special case of the problem of collinear collision between three balls, two of which are stationary prior to the impact, are of this type.

3. The third type of multiple impact problem, which can be called stochastic, is the most widely used one. It is characterized by high sensitivity of the result to the initial condition of the impact combined with the impossibility of determining these conditions with the necessary accuracy. In this case it is

impossible to find a unique correct solution of the multiple impact problem consistent with experiment. Our opinion is that one possible way of overcoming this paradox is to represent the solution as a random vector, i.e. a function which takes several values. Examples of this situation are considered above.

From the qualitative point of view, a stochastic system is analogous to a coin hitting a horizontal table and parallel to a vertical plane. Theoretically it should remain standing on its edge, but in practice we will have either heads or tails with the same probability.

A more complex situation arises for an impact involving more bodies. Here the impact impulse can take as large (but finite) a number of different values as desired.

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REFERENCES

1. BERNOULLI J., *Die Werke von Jakob Bernoulli*. Birkhäuser, Basel, 1969–1993.
2. MACLAURIN C., A treatise on fluxions. Ruddimans, Edinburgh, 1742.
3. D'ALEMBERT J., *Traite de Dynamique*. David, Paris, 1743.
4. IVANOV A. P., On impacts in a system with several unilateral constraints. *Prikl. Mat. Mekh.* **51**, 4, 559–566, 1987.
5. KOZLOV V. V. and TRESHCHEV D. V., *Billiards*. Izd. Moskov. Gos. Univ., Moscow, 1991.
6. GOLDSMIT V., *Impact. The Theory and Physical Behaviour of Colliding Solids*. Edward Arnold, London, 1960.
7. KIRGETOV V. I., On absolutely elastic impact in material systems. *Prikl. Mat. Mekh.* **24**, 5, 781–789.
8. KIL'CHEVSKII N. A., *Dynamical Contact Compression of Solids. Impact.* Naukova Dumka, Kiev, 1976.
9. LANKARANI H. M. and NIKRAVESH P. E., A contact force model with hysteresis damping for impact analysis of multibody systems. *Trans. ASME J. Mech. Design* **112**, 3, 369–376, 1990.
10. APPELL P., *Traité de Méchanique Rationelle*, Vol. 2. Gauthier-Villars, Paris, 1953.
11. CORIOLIS G., *Theorie Mathematique des Effets du Jeu de Billiard*. Carilian-Goeuvy, Paris, 1835.
12. NAGAYEV R. F. and KHOLODILIN N. A., On the theory of billiard ball collisions. *Izv. Ross. Akad. Nauk, MTT* 6, 48–55, 1992.

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